(1) Find the sylow subgroups of $D_{12} \times \mathbb{Z}_2$

(a) How many are there?

Let n_p = number of Sylow p-subgroups

$\underline{S3}$ $n_p \,\big|\, |G| \iff n_p \mid 24 \iff n_p \mid 2^3 \cdot 3$

$n_p \equiv 1 \mod p$

$\underline{n_p = 2}$

$$\left.\begin{array}{l} n_2 \mid 24 \\ n_2 \equiv 1 \mod 2 \end{array}\right\} \quad n_2 \in \{1, 3\}$$

$\underline{n_p = 3}$

$n_p \in \{1, 4\}$

Find the Sylow 3-subgroups

$D_{12} \times \{e\} \trianglelefteq D_{12} \times \mathbb{Z}_2$

as $[D_{12} \times \mathbb{Z}_2 : D_{12} \times \{e\}] = 2$

So we may find the sylow 3 subgroup of D_{12}

Let P be any Sylow 3-subgroups of D_{12}

$\underbrace{\{gPg^{-1}\}}$ (for $g \in D_{12} \times \mathbb{Z}_2$) are the set of all Sylow 3-subgroups of $D_{12} \times \mathbb{Z}_2$

these are by normality in $D_{12} \times \{e\}$

Find a Sylow 3-subgroup of D_{12}

(so P s.t. $P \leq D_{12}$ and $|P| = 3$)

eg $\langle (135)(246) \rangle$

D_{12} : $n_3 \in \{1, 4\}$ claim $n_3 = 1$ in D_{12}

Suppose not, else $n_3 = 4$

Say $P_1, P_2, P_3, P_4 \cong \mathbb{Z}_3$

$P_i \cap P_j = \{x\}$ if $i \neq j$. If $x = e$

If $x \neq e \Rightarrow P_i = P_j$

# of elements

$$\underbrace{1 + 4 \times 2}_{\substack{\text{elements of order} \\ 3 \text{ in } P_1, \ldots, P_4}} + \underbrace{1}_{6\text{-cycle}} + \underbrace{3}_{\substack{\text{reflections} \\ (\text{order } 2)}} + \underbrace{\tau^5}_{(\text{order } 6)} \geq 13 > 12$$

So we have a unique sylow

$\Rightarrow n_3 = 1$

$\Rightarrow$ Thus $P = \langle (123)(456) \rangle$ is the unique Sylow 3-subgroup of D_{12}

so $P \times \{e\}$ ———————— of $D_{12} \times \mathbb{Z}_2$

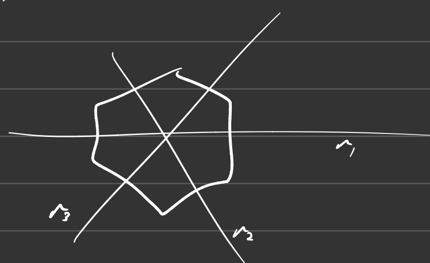
(2) Sylow 2-subgroups

$n_3 \in \{1, 3\}$

Note that in D_{12}, n_2 is also 1 or 3

We'll find the sylow 3-subgroups of D_{12}, say Q_1, Q_2, Q_3 then

$Q_i \times \mathbb{Z}_2$



Let $P_i = (r_i, \tau^3)$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$r = (12)(63)(45)$, $\tau^3 = (14)(25)(36)$

$r\tau^3 = (15)(25)$

$\Rightarrow Q_1 = \{e, r, \tau^3, r\tau^3\}$

$\Rightarrow W_{12} \times \mathbb{Z}_2$ are

$Q_1 \times \mathbb{Z}_2, \quad Q_2 \times \mathbb{Z}_2, \quad Q_3 \times \mathbb{Z}_2$

Q2 1029, 1536: Groups of ~~these~~ orders are not simple

<u>1029:</u> S3 $\Rightarrow$ normal Sylow 7-subgroups

<u>1536:</u> $= 2^9 \cdot 3$

$n_2 \equiv 1 \mod 2 \quad \Rightarrow$ only $n_2 = 1, n_2 = 3$ are possible as $n_2 \mid 2^9 \cdot 3$

Suppose $n_2 \neq 1 \Rightarrow n_2 = 3$

Let G act on the set of sylow 2-subgroups by conjugation

Get map $G \xrightarrow{\phi} S_3$

$\text{Kernel}(\phi) = \{e\}$ or $\underbrace{\text{Ker}(\phi) = G}$

If $\text{Ker}(\phi) = G$, then $gPg^{-1} = P$ for all $g \in G$, some Sylow 2-subgroup P

$\Rightarrow P$ is normal

$$G \simeq \frac{G}{\ker(f)} \simeq \underbrace{\mathrm{Im}(f)}_{S_3 \text{ or } \mathbb{Z}_3}$$

$\ker(f) = \{ g \in G \mid gPg^{-1} = P \quad \forall P$ in the set of Sylow 2-subgroups$\}$

<u>Claim</u>

S_4 is the unique subgroup of order 24 such that no Sylow p-subgroup is normal

$n_2 = 1$ or $3 \Rightarrow 3$ subgroups of order 8

$n_3 = 1$ or $4 \Rightarrow 4$ subgroups of order 3

<u>Use</u>: G acts on the set of Sylow 3 subgroups, say $\{P_1, P_2, P_3, P_4\}$

by conjugation, so $G \xrightarrow{f} S_4$ is a group hom

in fact such an action is always transitive so $\forall$ i,j there is

$g \in G$ st $g \cdot P_i = P_j$

Transitive subgroups of S_4

S_4 $(13) \cdot 1 = 3$

$(ab) \cdot a = b$

S_4, A_4, K_4, D_8, Z_4

$(abc) \cdot a = b$

$G \xrightarrow{f} S_4$

$\mathrm{Im}(f)$ transitive so

$\mathrm{Im}\, f \in \{ \cancel{S_4}, A_4, K_4, Z_4, \cancel{D_8} \}$

$N \trianglelefteq H$, $f^{-1}(N) \trianglelefteq G$

$G/\ker \xrightarrow{f} A_4$ Find $N \trianglelefteq A_4$, then $f^{-1}(N) \trianglelefteq G$

so if $|N| = 4 \Rightarrow f^{-1}(N) = 8$ is a Sylow 2-subgroup

Sylow 2-subgroups in A_4

$n_2 \mid 12 \Rightarrow 1, \cancel{2}, 3, \cancel{4}, \cancel{6}, \cancel{12}$

$n_2 \equiv 1 \bmod 2 \Rightarrow 1, 3$

Assume $n_2 = n_3$, get contradiction by
by ~~counting~~ elements

Assignment 3

Q1 Give an example of a group G, not nilpotent nor S_n st if $n \mid |G|$ then there exists $H \leq G$ st $|H| = n$

Ex

D_{2m} (order $2m$), Dic_{12}, ...

Q2 Suppose G, H abelian. Find the number of isomorphism classes of semi direct product groups which are abelian

$$S = \{ G \rtimes_{\alpha_1} H, G \rtimes_{\alpha_2} H, \ldots, H \rtimes_{\alpha_m} G, \ldots \} / \sim$$

$\alpha_1 : G \longrightarrow Aut(H)$
$\quad\; H \longrightarrow Aut(G)$

$u \sim u'$ iff isomorphic

How many elements of S are abelian?
At least one, namely $G \times H \simeq H \times G$

claim

no other semidirect product group is Abelian. Say $G \rtimes_\alpha H$ is

$$\left.\begin{aligned}(g,h)(g',h') &= \dots \\ (g',h')(g,h) &= \dots\end{aligned}\right\}$$ must agree for all g', h'

$\varphi_h(g') = g'\varphi_{h'}(g) \Rightarrow$ must hold for $h' = e$ $g = e$

$\Rightarrow \varphi_h(g) = g \quad \forall g \Rightarrow \varphi_h = id$

Q3 $Aut(D_{\mathbb{Z}_{12}}) \cong D_{12}$ $(|D_{12}| = 12)$

Fact

$D_{\mathbb{Z}_{12}} = \{a^i, xa^j \mid 0 \le i \le 5, 0 \le j \le 5\} = \langle x, a \rangle$

group operation in $D_{\mathbb{Z}_{12}}$

$(a^i)(a^j) = a^{i+j}$

$x^2 = a^3$

$xa = a^{-1}x$

$(xa^i)(a^j) = xa^{i+j}$

$(xa^i)(xa^j) = a^{3+j-i}$

$(a^i)(xa^j) = xa^{j-i}$

g	$a^0 = e$	a^1	a^2	a^3	a^4	a^5
$\lvert g \rvert$	1	6	3	2	3	6

g	xa^0	xa	xa^2	xa^3	xa^4	xa^5
$\|g\|$	4	4	4	4	4	4

Let $f: D_{12} \longrightarrow D_{12}$

assume it preserves orders, then

$$f: \begin{array}{l} a \longmapsto a \text{ or } a \\ x \longmapsto xa^i \end{array} \qquad (0 \leq i \leq 5)$$

$$f(\underbrace{a^{i_1} x^{j_1} a^{i_2} x^{j_2} \ldots a^{i_n} x^{j_n}}_{\in D_{12}}) = f(a)^{i_1} f(x)^{j_1} \ldots f(x)^{j_n}$$

Let $f_{\pm k} = \begin{cases} a \longmapsto a^{\pm} \\ x \longmapsto xa^k \end{cases}$

Example: $f_{-2}: \begin{array}{l} a \longmapsto a^{-1} \\ x \longmapsto xa^2 \end{array}$

<u>claim</u>: all $f_{\pm i}$ are automorphisms (check homomorphism)

$$f_{\pm k}(xa^i xa^j) = f_{\pm k}(xa^i) f_{\pm k}(xa^j) \ ?$$

$$\begin{array}{ll} \| & \| \\ f_{\pm k}(a^{3+j-i}) & \underbrace{(xa^{k\pm i})(xa^{k\pm j})} \\ \| & \\ a^{\pm(3+j-i)} & a^3 a^{k\pm j - k \mp j} = a^{3 \pm j \mp i} \end{array}$$

$\mathrm{Aut}(\mathrm{Dic}_{12}) = \{ f_{+i}, f_{-j} \mid 0 \le i \le 5, 0 \le j \le 5 \}$

order 12 !

A_4, Dic_{12}, D_{12}

<u>Identities</u>

$f = f_{+1}$

$a \xmapsto{f} a \mapsto \dots \mapsto a$

$x \xmapsto{f} xa \xmapsto{f} xaa \xmapsto{f} xa^2 \xmapsto{f} xa^5 \mapsto xa^6 = x$

$\Rightarrow |x| = 6$

~~$f_{-1} = g$~~

$|f_{-1}| = 2$; $a \xmapsto{g} a^{-1} \xrightarrow{g} a$

$x \mapsto xa \mapsto xaa^{-1} = O$

<u>$g = f_{-2}$</u>

$a \xmapsto{g} a^{-1} \xmapsto{g} a$

$x \mapsto xa^2 \mapsto (xa^2 a^{-2}) = x$

$|f_{-2}| = 2$

$\Rightarrow \mathrm{Aut}(\mathbb{Z}/\mathbb{Z}_{12}) \cong D_{12}$

$D_{12} = \langle r, \bar{\tau} \rangle$

$f_{+1} \longmapsto \bar{\tau}$

$f_{-1} \longmapsto r$

Assignment 4

Q1 Prove that if $q | p-1$ (use semi-direct product) then a non-Abelian group of order pq exists

Solution

~~"Let G be a non Abelian group of order pq"~~

Cannot use existence of group to prove the existence of it

$Aut(C_p) \cong C_{p-1}$ ($|Aut(C_p)| = p-1$ is sufficient)

For this, consider $x \xmapsto{\varphi_k} x^k$ where $\langle x \rangle = C_p$

$\varphi_k \circ \varphi_l = \varphi_{kl \bmod p}$

$\Rightarrow Aut(C_p) \cong (\mathbb{Z}_p)^*$

$\varphi_k \mapsto [k]$

Since $q | p-1$ and q is a prime

$\exists\ H \leq Aut(C_p)$ st $|H| = q$

and $H = \langle \varphi \rangle$

$|C_p \rtimes C_q| = pq$ (by construction)

If we find a non trivial map

$C_q \longrightarrow Aut(C_p)$, say φ, then

by Assignment 3 Q2

$C_p \rtimes_\varphi C_q$ is non-Abelian

Let $C_q = \langle y \rangle$,

let $\varphi : C_q \to Aut(C_p)$ by $\varphi(y) = \phi$

$\Rightarrow Im\,\varphi = H \leq Aut(C_p)$

$\Rightarrow C_p \rtimes_\varphi C_q$ is the desired group

Q2 $S_3 \cong \langle x, y \mid x^2, y^3, (xy)^2 \rangle = G$

$G = F(x,y)/N$, N = normal closure of $\{x^2, y^3, xyxy\} \leq F(x,y)$

proof

(1) Epimorphism $G \overset{\varphi}{\twoheadrightarrow} S_3$

(2) Epimorphism is an isomorphism

Step (1)

Let $S = \{x, y\}$,

define $\varphi : S \to S_3$ $\quad x \mapsto (12)$
$\quad y \mapsto (123)$

By FTGP

$\tilde{\varphi} : F(x,y)/N \; (= G) \longrightarrow S_3$

if $\bar{\varphi}(r_i) = e \in S_3$ for r_i relators

$\bar{\varphi} : F(S) \to S_3$

check $\overline{\varphi}(x^2) = \overline{\varphi}(x)\overline{\varphi}(x) = \varphi(x)\varphi(x)$

$= (12)(12) = e \in S_3$

$\overline{\varphi}(y^3) = \overline{\varphi}(y)\overline{\varphi}(y)\overline{\varphi}(y) = \ldots$

$\ldots = (123)^3 = e \in S_3$

$\overline{\varphi}((xy)^2) = e \in S_3$

$\Rightarrow$ this a homomorphism $G \to S_3$,

but $\langle \varphi(x), \varphi(y) \rangle = S_3$ so its surjective

Step 2 : φ is isomorphism

$|G| \leq |S_3| = 6$

Draw a graph: vertices = cosets w N

edges = prefixing by x, y, x^{-1}, y^{-1}

If the graph is closed under these operations (so the vertices have edges, starting at the vertex, labelled x, x^{-1}, y, y^{-1}

$$\begin{tikzcd}
 & yw & \\
y^{-1}w \arrow[r] & w \arrow[u, "y"] \arrow[r] & xw \\
 & x^{-1}w \arrow[u, "x"] &
\end{tikzcd}$$

Then every element wN is represented by a vertex

(proof: follow the path for w starting at the identity)

$$xxN = N = 1N$$

$$yyyN = N = 1N$$

$$xyxyN = N$$

$$\Rightarrow \begin{tikzcd} w \arrow[r, "x"] & xw \arrow[l, bend left, "x"] \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} w \arrow[r, "y"] & yw \arrow[r, "y"] & yyw \arrow[ll, bend left, "y"] \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} yxyw \arrow[d, "x"'] & xyw \arrow[l] \\ xyxyw = w \arrow[r, "y"'] & yw \arrow[u, "x"'] \end{tikzcd} \qquad \square$$

$$\begin{tikzcd} x \arrow[d, bend left, "x"] \arrow[rr, bend left=60] & xy \arrow[l] \arrow[d, bend left, "x"] & xy^2 \arrow[l] \arrow[d, bend left, "x"] \\ 1 \arrow[u, bend left, "x"] \arrow[r, "y"'] & y \arrow[u, bend left, "x"] \arrow[r, "y"'] & y^2 \arrow[u, bend left, "x"] \arrow[ll, bend left=60, "y"] \end{tikzcd}$$

$\square \quad \square \quad \square$

$\Rightarrow |G| \le 6$

Let $\langle x_1, \dots, x_n \mid x_i x_j = x_{i+j \bmod n} \rangle \cong G$

(1) $G \twoheadrightarrow C_n$ $\qquad x_i = x^i \in C_n$

$$x_i x_j = x_{i+j \bmod n}$$

$$x^i x^j = x^{i+j \bmod n}$$

(2) φ is isomorphism

The relations force $x_j = (x_1)^j$

so $(x_1)^6 (x_1)^{-1} \in N$ = normal closure of those relations